§ 14.3 Partial Derivatives

ex1: $f(x,y) = x^2+3xy+y-1$; $P(4,-5)$

Def: $f_x(x,y) \equiv \frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$

$f_x(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0+h,y_0)-f(x_0,y_0)}{h}$

ex1: $f_x = \lim_{h\to 0} \frac{(x+h)^2+3(x+h)y+y-1-x^2-3xy-y+1}{h} =$

$\lim_{h\to 0} \frac{2xh+h^2+3yh}{h} = 2x+3y$

$f_x(4,-5) = -7$

$f_y = 3x+1$

$x^2+y^2=1$ ⊛

Diff ⊛ w.r. to x

$$2x+2y\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{-2x}{2y} \quad \text{if } y\neq 0$$

$$\frac{dy}{dx}=?$$

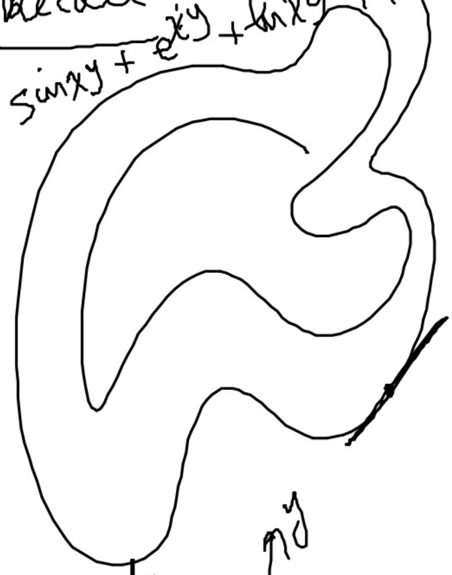
Implicit derivative

$y=x^2+1$

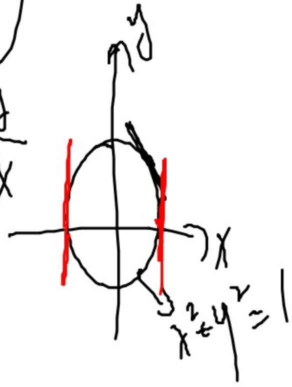
$y(x)$ explicit.

Recall

$$\sin xy + e^{xy} + \ln xy + 1 = 0$$



slope $=\frac{dy}{dx}$



ex 4 (*) $yz - \ln z = x + y$; $\frac{\partial z}{\partial x} = ?$

$z(x,y)$.

Diff. (*) w.r. to x to get

$$y\frac{\partial z}{\partial x} - \frac{1}{z}\frac{\partial z}{\partial x} = 1$$

$$\frac{\partial z}{\partial x}\left(y - \frac{1}{z}\right) = 1$$

$$\frac{\partial z}{\partial x} = \frac{z}{yz-1}$$

if $yz - 1 \neq 0$

at $(0,0,1)$ we get

$$-\frac{\partial z}{\partial x} = 1$$

$$\frac{\partial z}{\partial x}(0,0,1) = -1$$

Second derivatives:

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = 2$$

$$f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = 0$$

mixed deriv.

$$f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = 3$$

$$f_{yx} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 3$$

$$f(x,y) = x^2 + 3xy + y - 1$$

$$f_x = 2x + 3y$$

$$f_y = 3x + 1$$

Remark:

mixed derivatives need not be the same.

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$$xy + z^3 x - 2yz = 0 \qquad \frac{\partial z}{\partial x}(1,1,1) = ??$$

$z(x,y)$

diffs. all w.r. to x:

$$y + 3z^2 x \frac{\partial z}{\partial x} + z^3 - 2y\frac{\partial z}{\partial x} = 0$$

at $(1,1,1)$ we have:

$$1 + 3\frac{\partial z}{\partial x} + 1 - 2\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x}(1,1,1) = -2$$

30 $f(x,y,z) = yz \ln xy$

$f_x = yz \dfrac{1}{xy} y = \dfrac{yz}{x}$ (Note that $y \neq 0$)

$f_y = z \ln xy + yz \dfrac{1}{xy} x = z \ln xy + z$ (since $x \neq 0$)

$f_z = y \ln xy$

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$f_{xx} + f_{yy} + f_{zz} = 0$

$f_x = -\frac{1}{2}(2x)(x^2+y^2+z^2)^{-3/2} = -x(x^2+y^2+z^2)^{-3/2}$

$f_{xx} = -(x^2+y^2+z^2)^{-3/2} + 3x^2(x^2+y^2+z^2)^{-5/2}$; Similarly

$f_{yy} = -(x^2+y^2+z^2)^{-3/2} + 3y^2(x^2+y^2+z^2)^{-5/2}$

$f_{zz} = -(x^2+y^2+z^2)^{-3/2} + 3z^2(x^2+y^2+z^2)^{-5/2}$

$f_{xx} + f_{yy} + f_{zz} = -3(x^2+y^2+z^2)^{-3/2} + 3(x^2+y^2+z^2)^{-5/2}(x^2+y^2+z^2)^1$

$= -3(x^2+y^2+z^2)^{-3/2} + 3(x^2+y^2+z^2)^{-3/2} = 0$

$\underline{7}$ $f(x,y) = 1 - x + y - 3x^2y$; $f_x(1,2) = ??$ $f_y(1,2) = ??$

$$\frac{\partial f}{\partial y}(1,2) = \lim_{h \to 0} \frac{f(1, 2+h) - f(1,2)}{h}$$

$$= \lim_{h \to 0} \frac{1 - 1 + 2 + h - 3(2+h) + 4}{h}$$

$$= \lim_{h \to 0} \frac{-2h}{h} = -2$$

$f_y = 1 - 3x^2$

$f_y(1,2) = -2.$

Cramer's Rule

$$a_1 x + b_1 y = c_1$$

$$a_2 x + b_2 y = c_2$$

if $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

$$y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

Eg

$\boxed{\begin{matrix} u(x,y) \\ v(x,y) \end{matrix}}$

$$v \ln u = x$$
$$u \ln v = y$$

Diff both equations w.r.t. x:

$$v_x \ln u + \frac{v}{u} u_x = 1 \quad (1)$$
$$u_x \ln v + \frac{u}{v} v_x = 0 \quad (2)$$

Solve

$$\ln u \, v_x + \frac{v}{u} u_x = 1$$
$$\frac{u}{v} v_x + \ln v \, u_x = 0$$

$$v_x = \frac{\begin{vmatrix} 1 & v/u \\ 0 & \ln v \end{vmatrix}}{\begin{vmatrix} \ln u & \frac{v}{u} \\ \frac{u}{v} & \ln v \end{vmatrix}} = \frac{\ln v}{\ln u \ln v - 1}$$

if $\neq 0$

$$u_x = \frac{\begin{vmatrix} \ln u & 1 \\ \frac{u}{v} & 0 \end{vmatrix}}{\ln u \ln v - 1} = \frac{-u/v}{\ln u \ln v - 1}$$

$\underline{43}$ $g(x,y) = x^2y + \cos y + y\sin x$

$g_x = 2xy + y\cos x$

$g_y = x^2 - \sin y + \sin x$

$g_{xx} = 2y - y\sin x$

$g_{xy} = 2x + \cos x$

$g_{yx} = 2x + \cos x$

$g_{yy} = -\cos y$